

ESSENTIAL CURVES IN HANDLEBODIES AND TOPOLOGICAL CONTRACTIONS

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ABSTRACT. If X is a compact set, a *topological contraction* is a self-embedding f such that the intersection of the successive images $f^k(X)$, $k > 0$, consists of one point. In dimension 3, we prove that there are smooth topological contractions of the handlebodies of genus ≥ 2 whose image is essential. Our proof is based on an easy criterion for a simple curve to be essential in a handlebody.

1. INTRODUCTION

For a compact set X and a topological embedding $f : X \rightarrow X$, we shall say that f is a *topological contraction* if $\bigcap_{k \geq 0} f^k(X)$ consists of one point. We shall show that such a contraction can be very complicated when X is a 3-dimensional handlebody. Namely, we have the following result for which some more classical definitions will be recalled thereafter.

Theorem A. *There exists a North-South diffeomorphism f of the 3-sphere S^3 and a Heegaard decomposition $S^3 = P_- \cup P_+$ of genus $g \geq 2$ with the following properties:*

- 1) $f|_{P_+}$ is a topological contraction;
- 2) $f(P_+)$ is essential in P_+ .

We shall limit ourselves to $g = 2$, since the generalization will be clear. We recall that a 3-dimensional *handlebody* of genus 2 is diffeomorphic to the regular neighborhood P in \mathbb{R}^3 of the planar figure eight Γ . A *compression disk* of P is a smooth embedded disk in P whose boundary lies in ∂P in which it is not homotopic to a point. Among the compression disks are the *meridian* disks $\pi^{-1}(x)$, where x is a regular point¹ in Γ and $\pi : P \rightarrow \Gamma$ is the regular neighborhood projection (that is, a submersion over the smooth part of Γ). A subset X of P_+ is said to be *essential* in P_+ if it intersects every compression disk².

A diffeomorphism f of S^3 is a North-South diffeomorphism if it has two fixed points only, one source $\alpha \in P_-$ and one sink $\omega \in P_+$, every other orbit going from α to ω .

A *Heegaard splitting* of S^3 is made of an embedded surface dividing S^3 into two handlebodies. According to a famous theorem of F. Waldhausen such a decomposition is unique up to diffeomorphism [4] (hence up to isotopy after Cerf's theorem $\pi_0(Diff_+ S^3) = 0$ [1]). It is not

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¹Any point other than the center of the figure eight.

²This definition goes back to Rolfsen's book [3] p. 110.

hard to prove that the phenomenon mentioned in theorem A does not happen with a Heegaard splitting of genus 1: if T is a solid torus and f is a topological contraction of T , then there is a compression disk of T avoiding $f(T)$.

The example which we are going to construct for proving theorem A is based on the next theorem, for which some more notation is introduced. Let $\Gamma_0 \subset \Gamma$ be a simple closed curve. There exists a solid torus $T \subset \mathbb{R}^3$ which contains P and which is a tubular neighborhood of Γ_0 . Let $i_0 : P \rightarrow T$ be this inclusion. We say that a simple curve is unknotted in T if it bounds an embedded disk in T .

Theorem B. *There exists an essential simple curve C in P such that $i_0(C)$ is unknotted in T .*

Theorem B looks very easy as it is simple to draw a simple curve which intuitively satisfies its conclusion. Nevertheless, it appears that there are very few criteria for proving that a curve is essential in P . We are going to give one which is not algebraic in nature. Question: does there exist a topological algebraic tool which plays the same role.

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2. ESSENTIAL CURVES

Our candidate for C in Theorem B is pictured in figure 1.

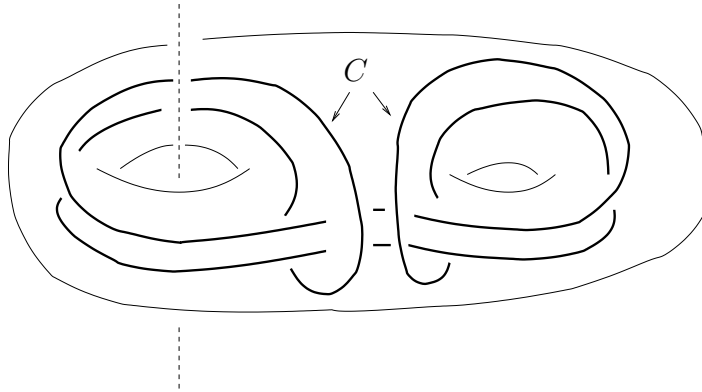


Figure 1

It is clear that $i_0(C)$ is unknotted in T (or, equivalently, in the complement of the vertical axis which is drawn on figure 1 and whose T is a compact retract by isotopy deformation). Instead of proving that C is essential in P , we are going to prove a stronger result. Clearly Proposition 1 below implies Theorem B.

³ Michel Boileau, Thomas Fiedler, John Guaschi and Claude Hayat.

Proposition 1. *Let $p : \tilde{P} \rightarrow P$ be the universal cover of P and let \tilde{C} be the preimage $p^{-1}(C)$. Then \tilde{C} is essential in \tilde{P} .*

Proof. We have the following description of \tilde{P} : it is a 3-ball with a Cantor set E removed from its bounding 2-sphere⁴. This Cantor set is the set of ends of \tilde{P} . A simple curve in $\partial\tilde{P}$ is not homotopic to zero if it divides E into two non-empty parts. We get a fundamental domain F for the action of $\pi_1(P)$ on \tilde{P} by cutting P along two non-parallel meridian disks D_0 and D_1 . Here is a description of $\tilde{C} \cap F$ (see figure 2): F is a 3-ball whose boundary consists of four disks d_0, d'_0, d_1, d'_1 and a punctured sphere $\partial_0 F$. We have $p(d_0) = p(d'_0) = D_0$ and $p(d_1) = p(d'_1) = D_1$. We have four strands in $\tilde{C} \cap F$: ℓ_1 and ℓ_2 joining d_0 and d_1 , ℓ'_0 (resp. ℓ'_1) whose end points belong to d'_0 (resp. d'_1). Moreover ℓ'_i , $i = 0, 1$, is linked with ℓ_j , $j = 1, 2$, in the following sense: any embedded surface whose boundary is made of ℓ'_i and a simple arc in d'_i intersects ℓ_j for $j = 1, 2$ (the algebraic intersection number is 1 for some choice of orientations).

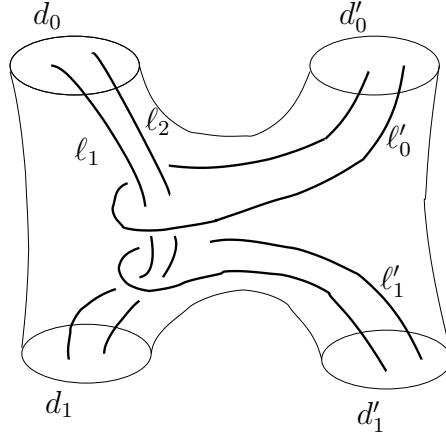


Figure 2

Globally \tilde{C} looks like an infinite Borromean chain: any finite number of components is unlinked. Suppose the contrary that \tilde{C} is not essential and consider Δ , a compression disk of \tilde{P} avoiding \tilde{C} . We take it to be transversal to $\tilde{D} := p^{-1}(D_0 \cup D_1)$. Let \mathcal{C} be the finite family of curves (arcs or closed curves) in $\tilde{D} \cap \Delta$. An element γ of \mathcal{C} is said to be *innermost* if γ divides Δ into two domains, one of them being a disk δ whose interior contains no element of \mathcal{C} . Take such an innermost element γ ; its associated disk δ lies in F , up to a covering transformation, and divides F into two balls F_0 and F_1 .

Lemma 1. *One of the balls, say F_0 , avoids \tilde{C} .*

Proof. Let us consider the case when $\gamma \subset d'_0$; say that $\ell'_0 \subset F_1$. The other cases are very similar. Let $\alpha = \delta \cap d'_0$. It is a simple arc dividing d'_0 into two parts. Both end points of ℓ'_0 lie in the same part since δ avoids ℓ'_0 . They are joined by a simple arc α' disjoint from α . Let δ' be an embedded disk bounded by $\ell'_0 \cup \alpha'$. This disk can be chosen disjoint from δ . Indeed, if

⁴Take the universal cover of Γ properly embedded in the hyperbolic plane and take a 3-dimensional thickening of it.

$\delta \cap \delta'$ is not empty, this intersection being transversal, by looking at an innermost intersection curve on δ one finds an embedded 2-sphere S in the complement of \tilde{C} with one hemisphere in δ and the other in δ' . As S bounds a 3-ball B_F in $\text{int } F$, which hence is also disjoint from \tilde{C} , there is an isotopy supported in a neighborhood of B_F whose effect on δ' decreases the number of intersection curves with δ .

Once $\delta \cap \delta'$ is empty, we have $\delta' \subset F_1$. But ℓ_1 and ℓ_2 must intersect δ' . Hence we have $\ell_1 \cup \ell_2 \subset F_1$. Similarly, we have $\ell'_1 \subset F_1$. \square

One checks easily that there is an isotopy of Δ , supported in a neighborhood of F_0 , till a new compression disk having fewer intersection curves with \tilde{D} than the cardinality of \mathcal{C} . Repeating this process, we push Δ into a fundamental domain, say F . In that position we have $\partial\Delta \subset \partial_0 F$. Again Δ divides F into two balls and one of them, F_0 , avoids \tilde{C} . This proves that $\partial\Delta$ bounds a disk in $\partial_0 F$, namely $F_0 \cap \partial_0 F$. Hence Δ is not a compression disk. \square

Remark. We used local linking information (namely, linking of strands in a fundamental domain of the universal covering space) which, as in this example, follows from usual linking numbers and we got a global result. This method looks very efficient. The general criterion is the following, where we use the same notation as above.

Criterion. *Let C be any simple closed curve in P . We assume that there is no embedded disk δ in F satisfying:*

- 1) *the boundary of δ is made of two arcs α and β , where α is an arc in \tilde{D} and β is an arc in $\partial\tilde{P} \cap F$;*
 - 2) *δ non trivially separates the components of $\tilde{C} \cap F$ (both components of $F \setminus \delta$ meet \tilde{C}).*
- Then C is essential in P .*

3. PROOF OF THEOREM A

We recall the embedding $i_0 : P \rightarrow \text{int } T$. We start with a curve C in P which meets the conclusion of Theorem B. We equip it with its 0-normal framing (a section in this framing is not linked with C in \mathbb{R}^3) and we choose an embedding $j_0 : T \rightarrow P$ whose image is a tubular neighborhood of C . Let B be a small ball in $\text{int } T$. As C is unknotted in T , there is an ambient isotopy, supported in $\text{int } T$, deforming i_0 to $i_1 : P \rightarrow \text{int } T$ such that $i_1 \circ j_0(T)$ is a standard small solid torus in B . One half of the desired Heegaard splitting of genus 2 will be given by $P_+ := i_1(P)$. At the present time f is only defined on T by $f := i_1 \circ j_0 : T \rightarrow \text{int } T$. If we compose i_1 with a sufficiently strong contraction of B into itself, then f is a contraction in the metric sense. Hence $\cap_{k>0} f^k(T)$ consists of one point.

Choose a round ball B' containing T in its interior. Since $f|_T$ is isotopic to the inclusion $T \hookrightarrow \mathbb{R}^3$, f extends as a diffeomorphism $B' \rightarrow B$, and further as a diffeomorphism $S^3 \rightarrow S^3$. We are free to choose $f : S^3 \setminus B' \rightarrow S^3 \setminus B$ as we like. If we compose f^{-1} with a strong contraction of $S^3 \setminus B'$, the intersection $\cap_k f^{-k}(S^3 \setminus B')$ consists of one point. We now have a North-South diffeomorphism f of S^3 which induces a topological contraction of T . Since

$f(T) \subset \text{int } P_+ \subset P_+ \subset \text{int } T$, f also induces a topological contraction of P_+ .

It remains to prove that $f(P_+)$ is essential in P_+ . We know that $i_1(C)$ is essential in P_+ . As a consequence, any compression disk Δ of P_+ crosses $f(T)$. We can take Δ to be transversal to $f(\partial T)$ such that no intersection curve is null-homotopic in $f(\partial T)$. Let γ be an intersection curve which is *innermost* in Δ and let δ be the disk that γ bounds in Δ .

Lemma 2. *We have $\delta \subset f(T)$.*

Proof. If not, we have $\delta \subset P_+ \setminus f(\text{int } T)$ and the simple curve γ in $f(\partial T)$ is unlinked with the core $i_1(C)$. Therefore, up to isotopy in $f(\partial T)$, it is a section of the 0-framing. In that case, $i_1(C)$ itself bounds an embedded disk in P_+ . This is impossible, as $i_1(C)$ is essential in P_+ . \square

Therefore δ is a compression disk of the solid torus $f(T)$. But $P_+ = i_1(P)$, like P itself, is essential in T . Hence $f(P_+)$ is essential in $f(T)$ and δ must cross $f(P_+)$. \square

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